

# ON A THEOREM OF SCHOEN AND SHKREDOV ON SUMSETS OF CONVEX SETS

LIANGPAN LI

ABSTRACT. A set of reals  $A = \{a_1, \dots, a_n\}$  labeled in increasing order is called convex if there exists a continuous strictly convex function  $f$  such that  $f(i) = a_i$  for every  $i$ . Given a convex set  $A$ , we prove

$$|A + A| \gg \frac{|A|^{14/9}}{(\log |A|)^{2/9}}.$$

Sumsets of different summands and an application to a sum-product-type problem are also studied either as remarks or as theorems.

## 1. INTRODUCTION

Let  $A = \{a_1, \dots, a_n\}$  be a set of real numbers labeled in increasing order. We say that  $A$  is **convex** if there exists a continuous strictly convex function  $f$  such that  $f(i) = a_i$  for every  $i$ . Hegyvári ([10]), confirming a conjecture of Erdős, proved that if  $A$  is convex then

$$|A - A| \gg |A| \cdot \frac{\log |A|}{\log \log |A|},$$

where “ $\gg$ ” is the Vinogradov notation. This result was later improved by many authors, see for example [5, 8, 9, 11, 14, 23] for related results. Recently, Schoen and Shkredov ([21]), combining an energy-type equality ([20])

$$(1.1) \quad E_3(A) = \sum_s E(A, A \cap (A + s)),$$

a useful set inclusion relation (see e.g. [13, 17, 18, 19, 20])

$$(1.2) \quad |(A + A) \cap (A + A + s)| \geq |A + (A \cap (A + s))|,$$

and an application (see Lemma 2.1 below) of the Szemerédi-Trotter incidence theorem (see e.g. [12, 24, 25]), proved for convex sets the following best currently known lower bounds:

$$(1.3) \quad |A + A| \gg \frac{|A|^{14/9}}{(\log |A|)^{2/3}},$$

$$(1.4) \quad |A - A| \gg \frac{|A|^{8/5}}{(\log |A|)^{2/5}}.$$

We also remark that Solymosi and Szemerédi obtained a similar result for convex sets, establishing  $|A \pm A| \gg |A|^{1.5+\delta}$  for some universal constant  $\delta > 0$ .

---

2000 *Mathematics Subject Classification.* 11B75.

*Key words and phrases.* sumset, productset, convex set, energy, Szemerédi-Trotter theorem.

The purpose of this note is twofold. Firstly, we give a slight improvement of (1.3) as follows:

**Theorem 1.1.** *Let  $A$  be a convex set. Then*

$$(1.5) \quad |A + A| \gg \frac{|A|^{14/9}}{(\log |A|)^{2/9}}.$$

Secondly, and most importantly, we will address an application of the Schoen-Shkredov estimate to a sum-product-type problem. Erdős and Szemerédi ([7]) once conjectured that the size of either the sumset or the productset of an arbitrary set of the reals must be very large, see [22] for the best currently known result toward this conjecture and related references therein. Another type of problem than one can attack regarding sumset and productset is to assume either one is very small, then prove the other one is very large. Elekes and Ruzsa ([6], see also [16, 22]) proved that if the sumset of a set is very small, then its productset must be very large. On the other hand, if the productset of a set is very small, say for example  $|AA| \leq M|A|$ , then the best currently known lower bound for the size of its sumset ([4], see also [5, 16, 22]) only is  $|A + A| \geq C_M |A|^{3/2}$ .

Roughly speaking, we will show that a set with very small multiplicative doubling is a “convex” set. Consequently, we can derive the following improvement.

**Theorem 1.2.** *Suppose  $|AA| \leq M|A|$ . Then*

$$\begin{aligned} |A + A| &\gg_M \frac{|A|^{14/9}}{(\log |A|)^{2/9}}, \\ |A - A| &\gg_M \frac{|A|^{8/5}}{(\log |A|)^{2/5}}. \end{aligned}$$

We remark that one can find direct application of Theorem 1.2 to the main result in [15], in which multi-fold sums from a set with very small multiplicative doubling are studied. See also [1, 2, 3] for some related discussions on multi-fold sumsets.

We collect some notations used throughout this note. Denote by  $\delta_{A,B}(s)$  the number of representations of  $s$  in the form  $a - b$ ,  $a \in A$ ,  $b \in B$ . If  $A = B$  we write  $\delta_A(s) = \delta_{A,A}(s)$  for simplicity. Furthermore, put

$$E(A, B) = \sum_s \delta_A(s) \delta_B(s) = \sum_s \delta_{A,B}(s)^2$$

and

$$E_k(A) = \sum_s \delta_A(s)^k.$$

Let  $A_s = A \cap (A + s)$ . All logarithms are to base 2. All sets are finite subsets of real numbers.

## 2. CONVEXITY AND ENERGY ESTIMATES

**Lemma 2.1** ([21]). *Let  $A$  be a convex set. Then for any set  $B$  and any  $\tau \geq 1$  we have*

$$|\{x \in A - B : \delta_{A,B}(x) \geq \tau\}| \ll \frac{|A| \cdot |B|^2}{\tau^3}.$$

A special case of Lemma 2.1 for  $B = -A$  was established in [11]. As applications, we have the following two lemmas.

**Lemma 2.2** ([21]). *Let  $A$  be a convex set. Then  $E_3(A) \ll |A|^3 \cdot \log |A|$ .*

**Lemma 2.3.** *Let  $A$  be a convex set. Then for any set  $B$  we have  $E(A, B) \ll |A| \cdot |B|^{1.5}$ .*

*Proof.* Let  $\Delta \doteq \frac{E(A, B)}{2|A||B|}$  and we divide  $E(A, B)$  into two parts, one is

$$\sum_{s: \delta_{A, B}(s) < \Delta} \delta_{A, B}(s)^2,$$

which is obviously less than half of  $E(A, B)$ , thus results in the other part

$$\sum_{s: \delta_{A, B}(s) \geq \Delta} \delta_{A, B}(s)^2,$$

being bigger than half of  $E(A, B)$ . Therefore, by Lemma 2.1 and a dyadic argument,

$$\frac{E(A, B)}{2} \leq \sum_{s: \delta_{A, B}(s) \geq \Delta} \delta_{A, B}(s)^2 \ll \sum_{j \geq 1} \Delta^2 \cdot 2^{2j} \cdot \frac{|A| \cdot |B|^2}{\Delta^3 \cdot 2^{3j}} \leq \frac{|A| \cdot |B|^2}{\Delta}.$$

This finishes the proof.  $\square$

**Lemma 2.4.** *Let  $A, B$  be any sets. Then*

$$\sum_s E(A_s, B) \leq E_3(A)^{2/3} \cdot E_3(B)^{1/3}.$$

*Proof.* Note  $\delta_{A_s}(t) = \delta_{A_t}(s)$ , which in common is  $|A \cap (A + s) \cap (A + t) \cap (A + s + t)|$ . Thus

$$\begin{aligned} \sum_s E(A_s, B) &= \sum_s \sum_t \delta_{A_s}(t) \delta_B(t) = \sum_s \sum_t \delta_{A_t}(s) \delta_B(t) \\ &= \sum_t \sum_s \delta_{A_t}(s) \delta_B(t) = \sum_t \delta_A(t)^2 \delta_B(t) \\ &\leq \left( \sum_t \delta_A(t)^3 \right)^{2/3} \cdot \left( \sum_t \delta_B(t)^3 \right)^{1/3} = E_3(A)^{2/3} \cdot E_3(B)^{1/3}. \end{aligned}$$

This finishes the proof.  $\square$

**Lemma 2.5.** *Let  $A, B$  be any sets. Then*

$$E_{1.5}(A)^2 \cdot |B|^2 \leq \left( \sum_s E(A_s, B) \right) \cdot E(A, A + B).$$

*Proof.* By the Cauchy-Schwarz inequality,

$$|A_s|^{1.5} \cdot |B| \leq E(A_s, B)^{1/2} \cdot |A_s + B|^{1/2} \cdot |A_s|^{1/2}.$$

First summing over all  $s \in A - A$ , then applying Cauchy-Schwarz again gives

$$\begin{aligned} E_{1.5}(A)^2 \cdot |B|^2 &\leq \left( \sum_s E(A_s, B) \right) \cdot \left( \sum_s |A_s + B| \cdot |A_s| \right) \\ &\leq \left( \sum_s E(A_s, B) \right) \cdot \left( \sum_s |(A + B)_s| \cdot |A_s| \right) \\ &= \left( \sum_s E(A_s, B) \right) \cdot E(A, A + B), \end{aligned}$$

where the second inequality is due to the set inclusion relation  $A_s + B \subset (A + B)_s$ . This finishes the proof.  $\square$

## 3. PROOF OF THEOREM 1.1

This section is mainly devoted to the proof of Theorem 1.1. We first claim

$$E_2(A)^3 \ll |A|^3 \cdot E_{1.5}(A)^2,$$

which follows simply from (see also the proof of Lemma 2.3)

$$E_2(A) = \sum_{s: \delta_A(s) < \Delta} \delta_A(s)^2 + \sum_{s: \delta_A(s) \geq \Delta} \delta_A(s)^2 \ll \sqrt{\Delta} \cdot E_{1.5}(A) + \frac{|A|^3}{\Delta}.$$

Then applying Lemma 2.5 with  $B = A$ , Lemma 2.4 and Lemma 2.2, we get

$$\frac{|A|^{12}}{|A + A|^3} \leq E_2(A)^3 \ll |A|^3 \cdot |A| \cdot (\log |A|) \cdot |A| \cdot |A + A|^{3/2},$$

which is equivalent to

$$|A + A| \gg \frac{|A|^{14/9}}{(\log |A|)^{2/9}}.$$

This finishes the proof of Theorem 1.1.

**Remark 3.1.** Let  $A, B$  be convex sets. We remark that one can establish

$$(3.1) \quad |A \pm B|^9 \gg \frac{|A|^6 \cdot |B|^8}{(\log |A|)^{4/3} \cdot (\log |B|)^{8/3}}.$$

To this aim, it suffices to note

$$\begin{aligned} \frac{|A|^2 \cdot |B|^2}{|A \pm B|} &\leq E(A, B) = \sum_s \delta_A(s) \cdot \delta_B(s) \\ &\leq \left( \sum_s \delta_A(s)^{3/2} \right)^{2/3} \cdot \left( \sum_s \delta_B(s)^3 \right)^{1/3} \\ &= E_{1.5}(A)^{2/3} \cdot E_3(B)^{1/3}, \end{aligned}$$

then turning to Lemmas 2.2~2.5 to get the desired inequality.

**Remark 3.2.** Let  $A, B$  be convex sets. We remark that one can establish

$$(3.2) \quad |A - A|^2 \cdot |A \pm B|^3 \gg \frac{|A|^6 \cdot |B|^2}{(\log |A|)^{4/3} \cdot (\log |B|)^{2/3}}.$$

To this aim, it suffices to note from the Hölder inequality that

$$\frac{|A|^6}{|A - A|} \leq E_{1.5}(A)^2,$$

then turning to Lemmas 2.2~2.5 to get the desired inequality.

## 4. PROOF OF THEOREM 1.2

**Lemma 4.1.** *Let  $A$  be a set of the form  $f(Z)$ , where  $f$  is a continuous strictly convex function,  $|Z + Z| \leq M|Z|$ . Then for any set  $B$  and any  $\tau \geq 1$ ,*

$$|\{x \in A - B : \delta_{A,B}(x) \geq \tau\}| \ll M^3 \cdot \frac{|A| \cdot |B|^2}{\tau^3}.$$

*Proof.* Without loss of generality, we may assume that  $f$  is monotonically increasing, and  $1 \ll \tau \leq \min\{|A|, |B|\}$ . Let  $G(f)$  denote the graph of  $f$  in the plane. For any  $(\alpha, \beta) \in \mathbb{R}^2$ , put  $L_{\alpha, \beta} = G(f) + (\alpha, -\beta)$ . Define the pseudo-line system  $\mathcal{L} = \{L_{z, b} : (z, b) \in Z \times B\}$ , and the set of points  $\mathcal{P} = (Z + Z) \times (A - B)$ . By convexity,  $|\mathcal{L}| = |Z| \cdot |B| = |A| \cdot |B|$ . Let  $\mathcal{P}_\tau$  be the set of points of  $\mathcal{P}$  belonging to at least  $\tau$  curves from  $\mathcal{L}$ . By the Szemerédi-Trotter incidence theorem,

$$\tau \cdot |\mathcal{P}_\tau| \ll (|\mathcal{P}_\tau| \cdot |Z| \cdot |B|)^{2/3} + |Z| \cdot |B| + |\mathcal{P}_\tau|,$$

from which we can deduce (see also [21])

$$|\mathcal{P}_\tau| \ll \frac{|Z|^2 \cdot |B|^2}{\tau^3}.$$

Next, suppose  $\delta_{A, B}(x) \geq \tau$ . There exist  $\tau$  distinct elements  $\{z_i\}_{i=1}^\tau$  from  $Z$ ,  $\tau$  distinct elements  $\{b_i\}_{i=1}^\tau$  from  $B$ , such that  $x = f(z_i) - b_i$  ( $\forall i$ ). Now we define  $Z_i \triangleq z_i + Z$  ( $\forall i$ ) and  $\mathcal{M}_x(s) \triangleq \sum_{i=1}^\tau \chi_{Z_i}(s)$ , where  $\chi_{Z_i}(\cdot)$  is the characteristic function of  $Z_i$ . Since

$$(z_i + z, x) = (z_i, f(z_i)) + (z, -b_i) \in L_{z, b_i} \quad (\forall z, \forall i),$$

we have  $(s, x) \in \mathcal{P}_{\mathcal{M}_x(s)}$ . Obviously,

$$\sum_{s \in Z+Z} \mathcal{M}_x(s) = \sum_{i=1}^\tau \sum_{s \in Z+Z} \chi_{Z_i}(s) \geq \tau |Z|.$$

Thus by the standard popularity argument,

$$|\{s \in Z + Z : \mathcal{M}_x(s) \geq \frac{\tau}{2M}\}| \geq \frac{|Z|}{2}.$$

This naturally implies

$$|\{x \in A - B : \delta_{A, B}(x) \geq \tau\}| \cdot \frac{|Z|}{2} \leq |\mathcal{P}_{\frac{\tau}{2M}}|,$$

and consequently,

$$|\{x \in A - B : \delta_{A, B}(x) \geq \tau\}| \ll \frac{|\mathcal{P}_{\frac{\tau}{2M}}|}{|Z|} \ll M^3 \cdot \frac{|Z| \cdot |B|^2}{\tau^3} = M^3 \cdot \frac{|A| \cdot |B|^2}{\tau^3}.$$

This finishes the proof.  $\square$

It is rather easy to observe that, any property holds for convex sets in this note should also hold for sets of the form  $f(Z)$ , where  $f$  is a continuous strictly convex function,  $|Z + Z| \leq M|Z|$ , with  $\gg$  replaced by  $\gg_M$ .

As applications, let  $A$  be a finite set of positive real numbers with  $|AA| \leq M|A|$ . Then  $A = \exp(Z)$ ,  $Z = \ln A$ ,  $|Z + Z| = |AA| \leq M|A| = M|Z|$ . Consequently, (1.5) and (3.2) hold for such an  $A$ . This suffices to prove Theorem 1.2. We are done.

**Acknowledgements.** This work was supported by the NSF of China (11001174).

## REFERENCES

- [1] J. Bourgain, M.-C. Chang, *On the size of  $k$ -fold sum and product sets of integers*, J. Amer. Math. Soc. 17 (2003), 473–497.
- [2] M.-C. Chang, *The Erdős-Szemerédi problem on sum set and product set*, Ann. Math. 157 (2003), 939–957.

- [3] E. Croot, D. Hart, *h-fold sums from a set with few products*, SIAM J. Discrete Math. 24 (2010), 505–519.
- [4] Gy. Elekes, *On the number of sums and products*, Acta Arith. 81 (1997), 365–367.
- [5] Gy. Elekes, M. B. Nathanson, I. Z. Ruzsa, *Convexity and sumsets*, J. Number Theory 83 (1999), 194–201.
- [6] Gy. Elekes, I. Z. Ruzsa, *Few sums, many products*, Studia Sci. Math. Hungar. 40 (2003), 301–308.
- [7] P. Erdős, E. Szemerédi, *On sums and products of integers*, In: Studies in Pure Mathematics (Birkhauser, Basel, 1983), 213–218.
- [8] M. Z. Garaev, *On lower bounds for the  $L_1$ -norm of an exponential sum*, Math. Notes 68 (2000), 713–720.
- [9] M. Z. Garaev, K.-L. Kueh, *On cardinality of sumsets*, J. Aust. Math. Soc. 78 (2005), 221–226.
- [10] N. Hegyvári, *On consecutive sums in sequences*, Acta Math. Acad. Sci. Hungar. 48 (1986), 193–200.
- [11] A. Iosevich, S. Konyagin, M. Rudnev, V. Ten, *Combinatorial complexity of convex sequences*, Discrete Comput. Geom. 35 (2006), 143–158.
- [12] H. Kaplan, J. Matoušek, M. Sharir, *Simple proofs of classical theorems in discrete geometry via the Guth–Katz polynomial partitioning technique*, arXiv:1102.5391, 2011.
- [13] N. H. Katz, P. Koester, *On additive doubling and energy*, SIAM J. Discrete Math. 24 (2010), 1684–1693.
- [14] S. V. Konyagin, *An estimate of the  $L_1$ -norm of an exponential sum*, In: The Theory of Approximations of Functions and Operators: Abstracts of Papers of the International Conference Dedicated to Stechkin’s 80th Anniversary (in Russian), pp. 88–89, 2000.
- [15] L. Li, *Multi-fold sums from a set with few products*, arXiv:1106.6074, 2011.
- [16] L. Li, J. Shen, *A sum-division estimate of reals*, Proc. Amer. Math. Soc. 138 (2010), 101–104.
- [17] T. Sanders, *On a non-abelian Balog-Szemerédi-type lemma*, J. Aust. Math. Soc. 89 (2010), 127–132.
- [18] T. Sanders, *On Roth’s theorem on progressions*, Ann. Math. 174 (2011), 619–636.
- [19] T. Schoen, *Near optimal bounds in Freiman’s theorem*, Duke Math. J. 158 (2011) 1–12.
- [20] T. Schoen, I. D. Shkredov, *Additive properties of multiplicative subgroups of  $\mathbb{F}_p$* , Quart. J. Math., to appear.
- [21] T. Schoen, I. D. Shkredov, *On sumsets of convex sets*, Comb. Probab. Comput. 20 (2011), 793–798.
- [22] J. Solymosi, *Bounding multiplicative energy by the sumset*, Adv. Math. 222 (2009), 402–408.
- [23] J. Solymosi, *Sumas contra productos*, Gaceta de la Real Sociedad Matemática Española, Vol. 12, No. 4, 2009.
- [24] L. Székely, *Crossing numbers and hard Erdős problems in discrete geometry*, Comb. Probab. Comput. 6 (1997), 353–358.
- [25] E. Szemerédi, W. T. Trotter, *Extremal problems in discrete geometry*, Combinatorica 3 (1983), 381–392.

DEPARTMENT OF MATHEMATICAL SCIENCES, LOUGHBOROUGH UNIVERSITY, LE11 3TU, UK

*E-mail address*: liliangpan@gmail.com